Problem 1.50

(a) Let $\mathbf{F}_1 = x^2\hat{\mathbf{z}}$ and $\mathbf{F}_2 = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. Calculate the divergence and curl of $\mathbf{F}_1$ and $\mathbf{F}_2$. Which one can be written as the gradient of a scalar? Find a scalar potential that does the job. Which one can be written as the curl of a vector? Find a suitable vector potential.

(b) Show that $\mathbf{F}_3 = yz\hat{\mathbf{x}} + zx\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written both as the gradient of a scalar and as the curl of a vector. Find scalar and vector potentials for this function.

Solution

The divergence and curl of $\mathbf{F}_1 = \langle 0, 0, x^2 \rangle$ are respectively

$$\nabla \cdot \mathbf{F}_1 = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(x^2) = (0) + (0) + (0) = 0$$

$$\begin{aligned}
\nabla \times \mathbf{F}_1 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & x^2 \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(0)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial z}(0)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(0)\right] \\
&= \hat{\mathbf{x}}\left[(0) - (0)\right] - \hat{\mathbf{y}}\left[(2x) - (0)\right] + \hat{\mathbf{z}}\left[(0) - (0)\right] \\
&= -2x\hat{\mathbf{y}}.
\end{aligned}$$

Because the divergence is zero, there exists a vector potential function $\mathbf{A}_1$ such that $\mathbf{F}_1 = \nabla \times \mathbf{A}_1$.

$$x^2\hat{\mathbf{z}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

For simplicity, choose $A_x = 0$ and $A_y = x^3/3$ and $A_z = 0$: $\mathbf{A}_1 = (x^3/3)\hat{\mathbf{y}}$. The divergence and curl of $\mathbf{F}_2 = \langle x, y, z \rangle$ are respectively

$$\nabla \cdot \mathbf{F}_2 = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = (1) + (1) + (1) = 3$$

$$\begin{aligned}
\nabla \times \mathbf{F}_2 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right] \\
&= \hat{\mathbf{x}}\left[(0) - (0)\right] - \hat{\mathbf{y}}\left[(0) - (0)\right] + \hat{\mathbf{z}}\left[(0) - (0)\right] \\
&= \mathbf{0}.
\end{aligned}$$

Because the curl is zero, there exists a scalar potential function $-\phi_2$ such that $\mathbf{F}_2 = -\nabla\phi_2$.

$$\langle x, y, z\rangle = -\left\langle \frac{\partial \phi_2}{\partial x}, \frac{\partial \phi_2}{\partial y}, \frac{\partial \phi_2}{\partial z}\right\rangle$$

For simplicity, choose $\phi_2(x, y, z) = -x^2/2 - y^2/2 - z^2/2$. The divergence and curl of $\mathbf{F}_3 = \langle yz, zx, xy\rangle$ are respectively

$$\begin{aligned}
\nabla \cdot \mathbf{F}_3 &= \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(zx) + \frac{\partial}{\partial z}(xy) = (0) + (0) + (0) = 0 \\
\nabla \times \mathbf{F}_3 &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix} \\
&= \hat{\mathbf{x}}\left[\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(zx)\right] - \hat{\mathbf{y}}\left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right] + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}(zx) - \frac{\partial}{\partial y}(yz)\right] \\
&= \hat{\mathbf{x}}\left[(x) - (x)\right] - \hat{\mathbf{y}}\left[(y) - (y)\right] + \hat{\mathbf{z}}\left[(z) - (z)\right] \\
&= \mathbf{0}.
\end{aligned}$$

Because the divergence and curl of $\mathbf{F}_3$ are zero, there exist scalar and vector potential functions, $-\phi_3$ and $\mathbf{A}_3$, such that $\mathbf{F}_3 = -\nabla\phi_3$ and $\mathbf{F}_3 = \nabla \times \mathbf{A}_3$.

$$\langle yz, xz, xy\rangle = -\left\langle \frac{\partial \phi_3}{\partial x}, \frac{\partial \phi_3}{\partial y}, \frac{\partial \phi_3}{\partial z}\right\rangle$$

$$yz\,\hat{\mathbf{x}} + xz\,\hat{\mathbf{y}} + xy\,\hat{\mathbf{z}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

For simplicity, choose

$$\begin{aligned}
\phi_3(x, y, z) &= -xyz \\
\mathbf{A}_3 &= \frac{1}{2}xz^2\hat{\mathbf{x}} + \frac{1}{2}x^2y\hat{\mathbf{y}} + \frac{1}{2}y^2z\hat{\mathbf{z}}.
\end{aligned}$$